Highlights

Supplementary Material for Low Rank Properties for Synchronizing Microphones and Sources in Ad-Hoc Wireless Acoustic Sensor Network

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- Three new low-rank properties are introduced to exploit the low-rank structure information between the time of arrival and unknown timing information in microphones and sources, all supported by mathematical proofs.
- A combined low-rank approximation algorithm is introduced to find global solutions for synchronizing microphones and sources by estimating the microphones' start time and the sources' emission time.
- The proposed new rank properties can be adapted for use in other fields with similar synchronization challenges, such as radio signals, potentially benefiting a broad spectrum of applications in technology and engineering.

Supplementary Material for Low Rank Properties for Synchronizing Microphones and Sources in Ad-Hoc Wireless Acoustic Sensor Network

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Abstract

This supplementary material provides the proof for proposed three variants of low-rank property, and the form of the Jacobian matrix for proposed combined low-rank approximation method.

The proof for three variants of low rank property (LRP) is based on two parts:

1) the LRP presented by state-of-the-arts [1, 2, 3] (Eq. (11) in the main manuscript: $rank(\mathbf{D}+\mathbf{U}) \leq 3$).

2) the theory of linear algebra [4, 5, 6]. By defining three matrices $\mathbf{E} \in \mathbb{R}^{m\times n}$, $\mathbf{\Theta} \in \mathbb{R}^{n\times h}$ and $\mathbf{O} \in \mathbb{R}^{m\times h}$, and two column vectors $\theta \in \mathbb{R}^{n}$ and $\mathbf{o} \in \mathbb{R}^{m}$, then we can have the corresponding linear algebra theorem [4, 5, 6]:

Theorem 1: Given one linear system $\mathbf{E}\theta = \mathbf{o}$, with coefficient matrix $\mathbf{E}$, augmented matrix $\begin{bmatrix}\mathbf{E} & \mathbf{o}\end{bmatrix} \in \mathbb{R}^{m\times(n+1)}$ and unknown column vector $\theta \in \mathbb{R}^{n}$, the following two items are sufficient and necessary if $m \geq n$:

- If $rank(\mathbf{E}) = rank(\begin{bmatrix}\mathbf{E} & \mathbf{o}\end{bmatrix}) = n$, $\mathbf{E}\theta = \mathbf{o}$ has a unique solution, and vice versa.
- If $rank(\mathbf{E}) = rank(\begin{bmatrix}\mathbf{E} & \mathbf{o}\end{bmatrix}) < n$, $\mathbf{E}\theta = \mathbf{o}$ has multiple solutions, and vice versa.

Based on the *Theorem 1*, we can extend one linear system to multiple linear systems by replacing those two column vectors, θ and $\mathbf{o}$, with two matrices, $\mathbf{\Theta}$ and $\mathbf{O}$, respectively, then those two items in *Theorem 1* are also sufficient and necessary [6]. Next, Sections A1, A2, and A3 show the proof of proposed LRPV1, LRPV2, and LRPV3, respectively. Finally, Section A4 provides the derivation of the Jacobian matrix for the proposed combined low-rank approximation (CLRA) method.

A1: Proof for Proposed LRPV1

This section shows the proof for the proposed LRPV1 in subsection A first, i.e., if $M-1 > N-1+3$, we shall prove

$$rank\left(\mathbf{T}_{\mathbf{1}}^{*}\right) \leq N-1+3, \tag{1}$$

where $\mathbf{T}_{\mathbf{1}}^{*} = \begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix} \in \mathbb{R}^{(M-1)\times 2(N-1)}$; $\mathbf{D} \in \mathbb{R}^{(M-1)\times(N-1)}$; $\mathbf{U} \in \mathbb{R}^{(M-1)\times(N-1)}$; M and N are the number of microphones and sources, respectively. Then, we show the details that result in $rank\left(\mathbf{T}_{\mathbf{1}}^{*}\right) < N-1+3$ or $rank\left(\mathbf{T}_{\mathbf{1}}^{*}\right) = N-1+3$ in subsection B.

A: Proof for LRPV1

From the LRP in state-of-the-arts [1, 2, 3]: $rank(\mathbf{D}+\mathbf{U}) \leq 3$, we can see that there exist 3 column vectors from matrix $\mathbf{D}+\mathbf{U}$ that could represent other column vectors of $\mathbf{D}+\mathbf{U}$ [1]. For the convenience of analysis, we assume the first 3 column vectors of $\mathbf{D}+\mathbf{U}$ are independent of each other, then we can also assume that there is an unknown matrix $\mathbf{X} \in \mathbb{R}^{3\times(N-1-3)}$ that enables the first 3 column vectors of matrix $\mathbf{D}+\mathbf{U}$ to represent the remaining column vectors of matrix $\mathbf{D}+\mathbf{U}$ [1, 3], i.e.,

$$(\mathbf{D}_{:,1:3} + \mathbf{U}_{:,1:3})\mathbf{X} = \mathbf{D}_{:,3+1:N-1} + \mathbf{U}_{:,3+1:N-1}. \tag{2}$$

Then, we can write Eq. (2) as the matrix multiplication form

$$\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix} \begin{bmatrix}\mathbf{X} \\ -\mathbf{I} \\ \mathbf{X}\end{bmatrix} = \mathbf{U}_{:,3+1:N-1}, \tag{3}$$

where $\mathbf{I} \in \mathbb{R}^{(N-1-3)\times(N-1-3)}$ is the identity matrix.

Since matrix $\begin{bmatrix} \mathbf{X} \\ -\mathbf{I} \\ \mathbf{X} \end{bmatrix} \in \mathbb{R}^{(N-1+3)\times(N-1-3)}$, the number of row for this matrix is $N-1+3$. In addition, we can also see that the coefficient matrix and augmented matrix in Eq. (3) are $\begin{bmatrix} \mathbf{D} & \mathbf{U}_{:,1:3} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$, respectively. Then based on the *Theorem 1* [4, 5, 6], we can have

$$rank(\begin{bmatrix} \mathbf{D} & \mathbf{U}_{:,1:3} \end{bmatrix}) = rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}) \leq N-1+3. \tag{4}$$

Now, we consider the conditions that lead matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ to be low-rank matrix. For one matrix, the number of both rows and columns for this matrix should be larger than the corresponding rank if such a matrix has low-rank property. Thus we need to consider two aspects, i.e., both number of columns and rows for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix} \in \mathbb{R}^{(M-1)\times 2(N-1)}$.

1) We consider the number of columns for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$, i.e., $2(N-1)$. Since $N-1>3$, we can have $2(N-1) > N-1+3$, which means that the number of columns for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ must be larger than the rank for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$, i.e., $N-1+3$.

2) We consider the number of rows for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$, i.e., $M-1$. From the conclusion of first aspect that the number of columns for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ is larger than the corresponding rank of matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ already, we can see that if the number of rows for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ is larger than the rank for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$, i.e., $M-1 > N-1+3$, matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ is a low-rank matrix. **This completes the proof for LRPV1** that $rank\left(\mathbf{T}_{\mathbf{1}}^{*}\right) \leq N-1+3$ if $M-1 > N-1+3$.

B: Details for LRPV1

From Eq. (4), we can see that $N-1+3$ provides a upper boundary of rank for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}$ if $M-1 > N-1+3$. Now, we analyze this upper boundary, i.e., the condition that leads $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}) < N-1+3$ or $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}) = N-1+3$. We divide the corresponding proof into two situations, i.e., $rank(\mathbf{D}+\mathbf{U}) < 3$ and $rank(\mathbf{D}+\mathbf{U}) = 3$.

Situation 1: If $rank(\mathbf{D}_{:,1:3} + \mathbf{U}_{:,1:3}) < 3$, we can have $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \end{bmatrix}) < N-1+3$.

Proof: From Eq. (3), we can see that sub-matrix I in Eq. (3) is identity matrix. Since the identity matrix is unique, we need to consider whether matrix $\mathbf{X}$ is unique or not. From the LRP [1, 2, 3], the corresponding relationship in Eq. (2) and the *Theorem 1* [4, 5, 6], it is obvious that matrix $\mathbf{X}$

has multiple solutions if $rank(\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}) = rank(\mathbf{D}+\mathbf{U}) < 3$, thus, with Eq. (3), we can have $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3$. **This completes the proof for *Situation 1.***

Situation 2: When $rank(\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}) = rank(\mathbf{D}+\mathbf{U}) = 3$, we can have three groups:

Group 1: If $rank(\mathbf{D}) < N-1$ or $rank(\mathbf{U}) < N-1$, it leads

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3. \tag{5}$$

Proof: We use the basic knowledge of linear algebra [6] to prove Eq. (5), i.e., multiplying all the elements of a certain column of the matrix by one same constant and then plus them to the corresponding elements of another column, it is obvious that the number of rank for the corresponding matrix remains unchanged [6]. Thus, if we multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}$ by 1, then plus them to the $\{j+N-1\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}$ where $j = 1, \quad \cdots, \quad N-1$, we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D} & \mathbf{D}+\mathbf{U}\end{bmatrix}). \tag{6}$$

Since there just three columns of matrix $\mathbf{D}+\mathbf{U}$ are independent of each other, i.e., $rank(\mathbf{D}+\mathbf{U}) = rank(\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}) = 3$, then we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{D}+\mathbf{U}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}). \tag{7}$$

For matrix $\begin{bmatrix}\mathbf{D} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ in Eq. (7), we can see that if $rank(\mathbf{D}) < N-1$, it means that there exists one column vector of matrix $\mathbf{D}$ can be represented by the remaining column vectors of matrix $\mathbf{D}$, resulting in

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}) < N-1+3. \tag{8}$$

With Eqs. (6), (7) and (8), we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3. \tag{9}$$

This completes the proof that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3$ ***if*** $rank(\mathbf{D}) < N-1$***.***

Similarity, if we multiply all of elements of $\{j+N-1\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}$ by 1, then plus them to the j^{th} column of matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}$ where $j = 1, \quad \cdots, \quad N-1$, we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D}+\mathbf{U} & \mathbf{U}\end{bmatrix}). \tag{10}$$

With Eq. (10), then follow the same steps as Eqs. (7), (8) and (9), it is easily prove that if $rank(\mathbf{U}) < N-1$, we have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3. \tag{11}$$

This completes the proof that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3$ ***if*** $rank(\mathbf{U}) < N-1$***.***

With Eqs. (9) and (11), we can conclude that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3$ if $rank(\mathbf{D}) < N-1$ or $rank(\mathbf{U}) < N-1$. ***This completes the proof for Group 1.***

Group 2: If $rank(\mathbf{D}) = rank(\mathbf{U}) = N-1$ and $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6$, it leads

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3. \tag{12}$$

Proof: First, we show one conclusion from the precondition that the rank of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix})$ is less than six, i.e., $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6$ in ***Group 2***. For any index $4 \leq n \leq N-1$, it is obvious that

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) \leq rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) + 1. \tag{13}$$

Since $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6$, we have $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) + 1 < 6+1 = 7$. Finally, from Eq. (13), we can conclude that

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 7. \tag{14}$$

Next, we use the contradiction method [7] to prove Eq. (12), i.e., we assume that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3$ is wrong, then from Eq. (4), we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3. \tag{15}$$

With Eq. (15), we have the following observation.

Observation: For any index $4 \leq n \leq N-1$, we have the observation that $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 7$.

Proof: With Eq. (15), then for any index $4 \leq n \leq N-1$, we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}) = 4+3 = 7. \tag{16}$$

By performing the elementary operation for the corresponding matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ in Eq. (16), i.e., multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ by 1, then plus them to the

$\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ where $j = 1, \cdots, 4$, then from Eq. (16), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3} & \mathbf{D}_{:,n}+\mathbf{U}_{:,n}\end{bmatrix}) = 7. \tag{17}$$

Since $\mathbf{D}_{:,n}+\mathbf{U}_{:,n}$ are dependent with $\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}$, from Eq. (17), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}) = 7. \tag{18}$$

Also, by performing the elementary operation for the corresponding matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ in Eq. (18), i.e., multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ by -1, then plus them to the $\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ where $j = 1, 2, 3$, then from Eq. (18), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 7. \tag{19}$$

This completes the proof for Observation.

On one side, Eq. (14) validates that $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 7$. On the other side, ***Observation*** shows that $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 7$ when $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3$, this causes a contradiction. Since Eq. (14) is correct, we can conclude $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3$ is wrong and have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3, \tag{20}$$

under the two following conditions that $rank(\mathbf{D}) = rank(\mathbf{U}) = N-1$ and $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6$. ***This completes the proof for Group 2.***

Group 3: If $rank(\mathbf{D}) = rank(\mathbf{U}) = N-1$ and $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6$, it leads

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3. \tag{21}$$

Proof: We also use the contradiction method [7] to prove Eq. (21) is correct, i.e., we assume $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3$ in Eq. (21) is wrong, then from Eq. (4), we can have

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+3. \tag{22}$$

With Eq. (22), we can have $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ or $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$. Next, we will show both $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ and $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$ are wrong with two steps.

Step 1: If $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ is correct, we have the following two observations.

Observation 1: The number of ranks for matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}$ is a function of $N-1$, i.e., $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}) = N-1+2$.

Proof: With both $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ and the statement from Eq. (4) that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix})$, it is obvious that

$$rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}) = N-1+2. \tag{23}$$

This completes the proof for Observation 1.

Observation 2: The number of ranks for matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}$ is constant which means it will not vary with $N-1$.

Proof: If $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$, it implies that for any index $4 \le n \le N-1$, we have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}) = 4+2 = 6 \tag{24}$$

Then do the elementary operation for matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ in Eq. (24), i.e., we can multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ by 1, then plus them to the $\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}$ where $j = 1, \ \cdots, \ 4$, we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3} & \mathbf{D}_{:,n}+\mathbf{U}_{:,n}\end{bmatrix}) = 6. \tag{25}$$

Since $\mathbf{D}_{:,n}+\mathbf{U}_{:,n}$ can be represented by $\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}$, i.e., $rank(\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}) = rank(\begin{bmatrix}\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3} & \mathbf{D}_{:,n}+\mathbf{U}_{:,n}\end{bmatrix})$, then from Eq. (25), we have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}) = 6. \tag{26}$$

Also, by performing the elementary operation for the following matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ in Eq. (26), i.e., multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ by -1, then plus them to the $\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ where $j = 1, \ 2, \ 3$, then from Eq. (26), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6. \tag{27}$$

From the precondition that $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6$ in ***Group 3*** and Eq. (27), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6. \tag{28}$$

Eq. (28) implies any $\mathbf{D}_{:,n}$ can be represented by $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}$. In addition, since $n \subset \{4, \ \cdots, \ N-1\}$, we have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6. \tag{29}$$

This completes the proof for Observation 2.

On one side, ***Observation 1*** shows that the number of ranks for matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}$ is a function of $N-1$, i.e., the number of ranks varies with $N-1$, On another side, ***Observation 2*** shows that the number of rank for matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}$ is a constant, meaning the number of rank for matrix $\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\end{bmatrix}$ will not vary with $N-1$, this causes a contradiction. ***Thus*** $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ ***is wrong***.

Step 2: If $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$, we have the following one observation.

Observation 3: $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6$.

Proof: If $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$, it implies for any $4 \leq n \leq N-1$, we have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3} & \mathbf{U}_{:,n}\end{bmatrix}) < 4+2 = 6. \tag{30}$$

By performing the elementary operation for the corresponding matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:4} & \mathbf{U}_{:,n}\end{bmatrix}$ in Eq. (30), i.e., multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:4} & \mathbf{U}_{:,n}\end{bmatrix}$ by 1, then plus them to the $\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:4} & \mathbf{U}_{:,n}\end{bmatrix}$ where $j = 1, \ \cdots, \ 4$, then from Eq. (30), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3} & \mathbf{D}_{:,n}+\mathbf{U}_{:,n}\end{bmatrix}) < 6. \tag{31}$$

Since $\mathbf{D}_{:,n}+\mathbf{U}_{:,n}$ are dependent with matrix $\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}$, in other words, $rank(\begin{bmatrix}\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3} & \mathbf{D}_{:,n}+\mathbf{U}_{:,n}\end{bmatrix}) = rank(\mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3})$, then from Eq. (31), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}) < 6. \tag{32}$$

Also, by performing the elementary operation for the corresponding matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ in Eq. (32), i.e., multiply all of elements of j^{th} column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ by -1, then plus them to the $\{j+4\}^{th}$ column of matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{D}_{:,1:3}+\mathbf{U}_{:,1:3}\end{bmatrix}$ where $j = 1, \ 2, \ 3$, then from Eq. (32), we can have

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6. \tag{33}$$

Since

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) \leq rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{D}_{:,n} & \mathbf{U}_{:,1:3}\end{bmatrix}), \tag{34}$$

then with Eq. (33), it implies

$$rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) < 6. \tag{35}$$

This completes the proof for Observation 3.

On one side, ***Observation 3*** shows that the number of rank for matrix $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix} < 6$ when $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$, on another side, the precondition in the ***Group 3*** shows that $\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix} = 6$, this causes a contradiction. ***Thus*** $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$ ***is wrong***.

In conclusion, ***Step 1*** validates that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+2$ is wrong and ***Step 2*** shows that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) < N-1+2$ is wrong, thus we can conclude that $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) \leq N-1+2$ is wrong and $rank(\begin{bmatrix}\mathbf{D} & \mathbf{U}\end{bmatrix}) = N-1+3$ is correct if $rank(\mathbf{D}) = rank(\mathbf{U}) = N-1$ and $rank(\begin{bmatrix}\mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3}\end{bmatrix}) = 6$. ***This completes the proof for Group 3.***

A2: Proof for Proposed LRPV2

This section shows the proof for proposed LRPV2 in subsection A first, i.e., if $N-1 > M-1+3$, we shall prove

$$rank\left(\mathbf{T}_{\mathbf{2}}^{*}\right) \leq M-1+3, \tag{36}$$

where $\mathbf{T}_{\mathbf{2}}^{*} = \begin{bmatrix}\mathbf{D}^{T} & \mathbf{U}^{T}\end{bmatrix} \in \mathbb{R}^{(N-1)\times 2(M-1)}$. Then, we show the details that lead $rank\left(\mathbf{T}_{\mathbf{2}}^{*}\right) < M-1+3$ or $rank\left(\mathbf{T}_{\mathbf{2}}^{*}\right) = M-1+3$ in subsection B.

A: Proof for LRPV2

From the LRP in state-of-the-arts [1, 2, 3]: $rank(\mathbf{D}+\mathbf{U}) = rank(\mathbf{D}^{T} + \mathbf{U}^{T}) \leq 3$, we can see that there exists 3 row vectors from matrix $\mathbf{D}+\mathbf{U}$ that could represent remaining row vectors of $\mathbf{D}+\mathbf{U}$ [1]. For the convenience of analysis, we assume the first 3 row vectors of matrix $\mathbf{D}+\mathbf{U}$ are independent from each other, then we can assume that there is an unknown matrix $\hat{\mathbf{X}} \in \mathbb{R}^{3\times(M-1-3)}$ that enables the first 3 row vectors of matrix $\mathbf{D}+\mathbf{U}$ to represent the other row vectors of matrix $\mathbf{D}+\mathbf{U}$, i.e.,

$$(\mathbf{D}_{1:3,:}^{T} + \mathbf{U}_{1:3,:}^{T})\hat{\mathbf{X}} = \mathbf{D}_{3+1:M-1,:}^{T} + \mathbf{U}_{3+1:M-1,:}^{T}. \tag{37}$$

Then, we can write Eq. (37) as the matrix multiplication form

$$\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T_{1:3,:}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ -\mathbf{I}\\ \hat{\mathbf{X}}\end{bmatrix} = \mathbf{U}^T_{3+1:M-1,:}, \tag{38}$$

where $\mathbf{I} \in \mathbb{R}^{(M-1-3)\times(M-1-3)}$ is the identity matrix.

Since matrix $\begin{bmatrix}\hat{\mathbf{X}}\\ -\mathbf{I}\\ \hat{\mathbf{X}}\end{bmatrix} \in \mathbb{R}^{(M-1+3)\times(M-1-3)}$, the number of rows for this matrix is $M-1+3$. In addition, we can also see the coefficient matrix and augmented matrix in Eq. (38) are $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T_{1:3,:}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$, respectively. Then based on the *Theorem 1* [4, 5, 6], we can have

$$rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T_{1:3,:}\end{bmatrix}) = rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) \leq M-1+3. \tag{39}$$

Now, we consider the conditions that lead matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ to be low-rank matrix. For one matrix, the number of both rows and columns for this matrix should be larger than the corresponding rank if such a matrix has low-rank property. Thus two aspects need to be considered, i.e., both number of columns and rows for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix} \in \mathbb{R}^{(N-1)\times 2(M-1)}$.

1) We consider the number of columns for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$, i.e., $2(M-1)$. Since $M-1>3$, we can have $2(M-1) > M-1+3$, which means that the number of columns for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ must be larger than the rank for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$, i.e., $M-1+3$.

2) We consider the number of rows for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$, i.e., $N-1$. From the conclusion of first aspect that the number of columns for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ is larger than the corresponding rank of matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ already, we can see that if the number of rows for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ is larger than the rank for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$, i.e., $N-1 > M-1+3$, matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ is a low-rank matrix. **This completes the proof for LRPV2** that $rank\,(\mathbf{T}_2^*) \leq M-1+3$ if $N-1 > M-1+3$.

B: Details for LRPV2

From Eq. (39), we can see that $M-1+3$ provides a upper boundary of rank for matrix $\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}$ if $N-1 > M-1+3$. Now, we analyze this upper boundary, i.e., the condition that leads $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) < M-1+3$ or $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) = M-1+3$. We can also divide the corresponding proof into two situations. ***Situation 1:*** If $rank(\mathbf{D}^T_{1:3,:} + \mathbf{U}^T_{1:3,:}) = rank(\mathbf{D}^T + \mathbf{U}^T) < 3$, we can have $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) < M - 1 + 3$. And ***Situation 2:*** If $rank(\mathbf{D}^T_{1:3,:} + \mathbf{U}^T_{1:3,:}) = rank(\mathbf{D}^T + \mathbf{U}^T) = 3$, we can also have three groups. ***Group 1:*** If $rank(\mathbf{D}^T) < M - 1$ or $rank(\mathbf{U}^T) < M - 1$, it leads $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) < M-1+3$. ***Group 2:*** If $rank(\mathbf{D}^T) = rank(\mathbf{U}^T) = M-1$ and $rank(\begin{bmatrix}\mathbf{D}^T_{1:3,:} & \mathbf{U}^T_{1:3,:}\end{bmatrix}) < 6$ it leads $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) < M-1+3$. ***Group 3:*** If $rank(\mathbf{D}^T) = rank(\mathbf{U}^T) = M - 1$ and $rank(\begin{bmatrix}\mathbf{D}^T_{1:3,:} & \mathbf{U}^T_{1:3,:}\end{bmatrix}) = 6$ it leads $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) = M-1+3$. Finally, by using the same proof procedure as LRPV1 in Section A1-B, it is easy to prove those two situations above. ***This completes the proof for the conditions that lead*** $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) < M - 1 + 3$ ***or*** $rank(\begin{bmatrix}\mathbf{D}^T & \mathbf{U}^T\end{bmatrix}) = M - 1 + 3$***.***

A3: Proof for Proposed LRPV3

This section shows the proof for the proposed LRPV3 in subsection A first, i.e.,

$$rank\left(\mathbf{T}^*_{\mathbf{3}}\right) \leq min(N - 1 + 3, M - 1 + 3), \tag{40}$$

where $\mathbf{T}^*_{\mathbf{3}} = \begin{bmatrix}\mathbf{D} & \mathbf{U}\\ \mathbf{U} & \mathbf{D}\end{bmatrix} \in \mathbb{R}^{2(M-1)\times 2(N-1)}$. Then, we show the details that lead $rank\left(\mathbf{T}^*_{\mathbf{3}}\right) < min(N-1+3, M-1+3)$ or $rank\left(\mathbf{T}^*_{\mathbf{3}}\right) = min(N-1+3, M-1+3)$ in subsection B.

A: Proof for LRPV3

We consider two situations for LRPV3, i.e., $M \geq N$ and $M < N$.

If $M \geq N$, the proof for LRPV3 in Eq. (40) is transformed to

$$rank\left(\begin{bmatrix}\mathbf{D} & \mathbf{U}\\ \mathbf{U} & \mathbf{D}\end{bmatrix}\right) \leq N - 1 + 3. \tag{41}$$

From Eq. (2), we can have

$$\begin{cases}\mathbf{D}_{:,1:3}\mathbf{X} - \mathbf{D}_{:,3+1:N-1} + \mathbf{U}_{:,1:3}\mathbf{X} = \mathbf{U}_{:,3+1:N-1}\\ \mathbf{U}_{:,1:3}\mathbf{X} - \mathbf{U}_{:,3+1:N-1} + \mathbf{D}_{:,1:3}\mathbf{X} = \mathbf{D}_{:,3+1:N-1}\end{cases}. \tag{42}$$

Then we can write Eq. (42) as matrix multiplication form

$$\begin{bmatrix}\mathbf{D} & \mathbf{U}_{:,1:3}\\ \mathbf{U} & \mathbf{D}_{:,1:3}\end{bmatrix}\begin{bmatrix}\mathbf{X}\\ -\mathbf{I}\\ \mathbf{X}\end{bmatrix} = \begin{bmatrix}\mathbf{U}_{:,3+1:N-1}\\ \mathbf{D}_{:,3+1:N-1}\end{bmatrix}, \tag{43}$$

where $\mathbf{I} \in \mathbb{R}^{(N-1-3)\times(N-1-3)}$ is the identity matrix.

Since matrix $\begin{bmatrix} \mathbf{X} \\ -\mathbf{I} \\ \mathbf{X} \end{bmatrix} \in \mathbb{R}^{(N-1+3)\times(N-1-3)}$, the number of rows for this matrix is $N-1+3$. In addition, we can see the coefficient matrix and the augmented matrix in Eq. (43) are $\begin{bmatrix} \mathbf{D} & \mathbf{U}_{:,1:3} \\ \mathbf{U} & \mathbf{D}_{:,1:3} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}$, respectively. Then based on the *Theorem 1* [4, 5, 6], we can have

$$rank(\begin{bmatrix} \mathbf{D} & \mathbf{U}_{:,1:3} \\ \mathbf{U} & \mathbf{D}_{:,1:3} \end{bmatrix}) = rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) \leq N-1+3. \tag{44}$$

This completes the proof for LRPV3 when $M \geq N$.

If $M < N$, the proof for LRPV3 in Eq. (40) is transformed to

$$rank\left(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}\right) \leq M-1+3. \tag{45}$$

From Eq. (37), we can have

$$\begin{cases} \mathbf{D}^T_{1:3,:}\hat{\mathbf{X}} - \mathbf{D}^T_{3+1:M-1,:} + \mathbf{U}^T_{1:3,:}\hat{\mathbf{X}} = \mathbf{U}^T_{3+1:M-1,:} \\ \mathbf{U}^T_{1:3,:}\hat{\mathbf{X}} - \mathbf{U}^T_{3+1:M-1,:} + \mathbf{D}^T_{1:3,:}\hat{\mathbf{X}} = \mathbf{D}^T_{3+1:M-1,:} \end{cases}. \tag{46}$$

Then we can write Eq. (46) as matrix multiplication form

$$\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T_{1:3,:} \\ \mathbf{U}^T & \mathbf{D}^T_{1:3,:} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}} \\ -\mathbf{I} \\ \hat{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} \mathbf{U}^T_{3+1:M-1,:} \\ \mathbf{D}^T_{3+1:M-1,:} \end{bmatrix}. \tag{47}$$

where $\mathbf{I} \in \mathbb{R}^{(M-1-3)\times(M-1-3)}$ is the identity matrix.

Since matrix $\begin{bmatrix} \hat{\mathbf{X}} \\ -\mathbf{I} \\ \hat{\mathbf{X}} \end{bmatrix} \in \mathbb{R}^{(M-1+3)\times(M-1-3)}$, the number of rows for this matrix is $M-1+3$. In addition, we can also see the coefficient matrix and the augmented matrix in Eq. (47) is $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T_{1:3,:} \\ \mathbf{U}^T & \mathbf{D}^T_{1:3,:} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}$, respectively. Then based on the *Theorem 1* [4, 5, 6], we can have

$$rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T_{1:3,:} \\ \mathbf{U}^T & \mathbf{D}^T_{1:3,:} \end{bmatrix}) = rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) \leq M-1+3. \tag{48}$$

Since $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix} = \begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}^T$, we can have

$$rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}^T) = rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) \leq M - 1 + 3. \tag{49}$$

This completes the proof for LRPV3 when $M < N$.

Finally, Based on Eq. (44) when $M \geq N$ and Eq. (49) when $M < N$, we can have

$$rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) \leq min(N - 1 + 3, M - 1 + 3). \tag{50}$$

This completes all the proof for LRPV3.

B: Details for LRPV3

We first show the details of LRPV3 in Eq. (44) when $M \geq N$ and then the details of LRPV3 in Eq. (48) are displayed when $M < N$.

(1) $M \geq N$: From Eq. (44), we can see that $N - 1 + 3$ provides a upper boundary of rank for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}$ if $M \geq N$. Now, we analyze this upper boundary, i.e., the condition that leads $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) < N - 1 + 3$ or $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) = N - 1 + 3$. We divide the corresponding proof into two situations. ***Situation 1:*** If $rank(\mathbf{D}_{:,1:3} + \mathbf{U}_{:,1:3}) = rank(\mathbf{D} + \mathbf{U}) < 3$, we can have $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) < M - 1 + 3$. ***Situation 2:*** If $rank((\mathbf{D}_{:,1:3} + \mathbf{U}_{:,1:3}) = rank(\mathbf{D} + \mathbf{U}) = 3$, we can have three groups. ***Group 1:*** If $rank(\mathbf{D}) < N - 1$ or $rank(\mathbf{U}) < N - 1$, it leads $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) < N - 1 + 3$. ***Group 2:*** If $rank(\mathbf{D}) = rank(\mathbf{U}) = N - 1$ and $rank(\begin{bmatrix} \mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3} \end{bmatrix}) < 6$ it leads $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) < N - 1 + 3$. ***Group 3:*** If $rank(\mathbf{D}) = rank(\mathbf{U}) = N - 1$ and $rank(\begin{bmatrix} \mathbf{D}_{:,1:3} & \mathbf{U}_{:,1:3} \end{bmatrix}) = 6$ it leads $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) = N - 1 + 3$. For the proof of these two situations, first, by performing the elementary operation for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}$, i.e., multiply all of elements of i^{th} row of matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}$ by 1, then plus them to the $\{i + M - 1\}^{th}$ column of matrix

$\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}$ where $i = 1, \cdots, M-1$, then we can have $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) = rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{D}+\mathbf{U} & \mathbf{U}+\mathbf{D} \end{bmatrix}) \leq N-1+3$. Then by using the same proof procedure as LRPV1 in Section A1-B for matrix $\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{D}+\mathbf{U} & \mathbf{U}+\mathbf{D} \end{bmatrix}$, it is easy to prove those two situations above. ***This completes the proof for the conditions that lead*** $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) < N-1+3$ ***or*** $rank(\begin{bmatrix} \mathbf{D} & \mathbf{U} \\ \mathbf{U} & \mathbf{D} \end{bmatrix}) = N-1+3$ ***when*** $M \geq N$***.***

(2) $M < N$: From Eq. (48), we can see that $M-1+3$ provides a upper boundary of rank for matrix $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}$ if $M < N$. Now, we analyze this upper boundary, i.e., the conditions that leads $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) < M-1+3$ or $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) = M-1+3$. We divide the corresponding proof into two situations. ***Situation 1:*** If $rank(\mathbf{D}^T_{1:3,:} + \mathbf{U}^T_{1:3,:}) = rank(\mathbf{D}^T + \mathbf{U}^T) < 3$, we can have $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) < M-1+3$. ***Situation 2:*** If $rank(\mathbf{D}^T_{1:3,:} + \mathbf{U}^T_{1:3,:}) = rank(\mathbf{D}^T + \mathbf{U}^T) = 3$, we can have three groups. ***Group 1:*** If $rank(\mathbf{D}^T) < M-1$ or $rank(\mathbf{U}^T) < M-1$, it leads $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) < M-1+3$. ***Group 2:*** If $rank(\mathbf{D}^T) = rank(\mathbf{U}^T) = M-1$ and $rank(\begin{bmatrix} \mathbf{D}^T_{1:3,:} & \mathbf{U}^T_{1:3,:} \end{bmatrix}) < 6$ it leads $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) < M-1+3$. ***Group 3:*** If $rank(\mathbf{D}^T) = rank(\mathbf{U}^T) = M-1$ and $rank(\begin{bmatrix} \mathbf{D}^T_{1:3,:} & \mathbf{U}^T_{1:3,:} \end{bmatrix}) = 6$ it leads $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) = M-1+3$. For the proof of these two situations, first, by performing the elementary operation for matrix $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}$, i.e., multiply all of elements of j^{th} row of matrix $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}$ by 1, then plus them to the $\{j+N-1\}^{th}$ column of matrix $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}$ where $j = 1, \cdots, N-1$, then from Eq. (44), we can have $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) =$

$rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{D}^T+\mathbf{U}^T & \mathbf{U}^T+\mathbf{D}^T \end{bmatrix}) \leq M-1+3$. Then by following the same proof procedure as LRPV1 in Section A1-B for matrix $\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{D}^T+\mathbf{U}^T & \mathbf{U}^T+\mathbf{D}^T \end{bmatrix}$, it is easy to prove those two situations above when $M < N$. ***This completes the proof for the conditions that lead*** $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) < M-1+3$ ***or*** $rank(\begin{bmatrix} \mathbf{D}^T & \mathbf{U}^T \\ \mathbf{U}^T & \mathbf{D}^T \end{bmatrix}) = M-1+3$ ***when*** $M < N$***.***

A4: Form of the Jacobian matrix for combined low-rank approximation Method

This section shows the form of the Jacobian matrix [8] of the proposed CLRA method: $\mathbf{J} = \partial\mathbf{q}/\partial\mathbf{p}$ where column vector $\mathbf{p} = \left[\dot{\delta}^T, \dot{\eta}^T, \mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T, \mathbf{w}^T\right]^T \in \mathbb{R}^P$; $\dot{\delta} = \begin{bmatrix}\dot{\delta}_1 & \cdots & \dot{\delta}_M\end{bmatrix}^T$; $\dot{\eta} = \begin{bmatrix}\dot{\eta}_1 & \cdots & \dot{\eta}_N\end{bmatrix}^T$; $P = M+N-1+3(N-1-3)+M_N(2(N-1)-M_N)+(N-1+3)(N-1-3)+(M-1+3)(M-1-3)$; $M_N = min(M-1+3, N-1+3)$ and

$$\begin{cases} \mathbf{x} = v(\mathbf{X}) \\ \mathbf{y} = v(\mathbf{Y}) \\ \mathbf{z} = v(\mathbf{Z}) \\ \mathbf{w} = v(\mathbf{W}) \end{cases},$$

and $v(\cdot)$ denotes operation for column-wise matrix vectorization; and the size of the following matrices are $\mathbf{X} \in \mathbb{R}^{3\times(N-1-3)}$; $\mathbf{Y} \in \mathbb{R}^{M_N\times(2(N-1)-M_N)}$; $\mathbf{Z} \in \mathbb{R}^{(N-1+3)\times(N-1-3)}$; $\mathbf{W} \in \mathbb{R}^{(M-1+3)\times(M-1-3)}$ and column vector $\mathbf{q} = \begin{bmatrix}\mathbf{f_A}^T & \lambda\mathbf{f_B}^T & \gamma\mathbf{f_C}^T & \alpha\mathbf{f_D}^T & \beta\mathbf{f_E}^T\end{bmatrix}^T \in \mathbb{R}^Q$,

$$\begin{cases} \mathbf{f_A} = v(\mathbf{U}) \\ \mathbf{f_B} = v((\mathbf{A}+\mathbf{F})\mathbf{X}-\mathbf{B}-\mathbf{G}) \\ \mathbf{f_C} = v(\mathbf{T^*_{31}}\mathbf{Y}-\mathbf{T^*_{32}}) \\ \mathbf{f_D} = v(\mathbf{T^*_{11}}\mathbf{Z}-\mathbf{T^*_{12}}) \\ \mathbf{f_E} = v(\mathbf{T^*_{21}}\mathbf{W}-\mathbf{T^*_{22}}) \end{cases}, \tag{51}$$

and $Q = (M-1)(8(N-1)-2M_N-6)-3(N-1)$; $\mathbf{A} = \mathbf{D}_{:,1:3} \in \mathbb{R}^{(M-1)\times 3}$; $\mathbf{B} = \mathbf{D}_{:,3+1:N-1} \in \mathbb{R}^{(M-1)\times(N-1-3)}$; $\mathbf{F} = \mathbf{U}_{:,1:3} \in \mathbb{R}^{(M-1)\times 3}$; $\mathbf{G} = \mathbf{U}_{:,3+1:N-1} \in$

$\mathbb{R}^{(M-1)\times(N-1-3)}$; $\mathbf{T_{11}^*} = \mathbf{T^*_{1:,1:N-1+3}} \in \mathbb{R}^{(M-1)\times(N-1+3)}$; $\mathbf{T_{12}^*} = \mathbf{T^*_{1:,N+3:2(N-1)}}$ $\in \mathbb{R}^{(M-1)\times(N-1-3)}$; $\mathbf{T_{21}^*} = \mathbf{T^*_{2:,1:M-1+3}} \in \mathbb{R}^{(N-1)\times(M-1+3)}$; $\mathbf{T_{22}^*} = \mathbf{T^*_{2:,M+3:2(M-1)}}$ $\in \mathbb{R}^{(N-1)\times(M-1-3)}$; $\mathbf{T_{31}^*} = \mathbf{T^*_{3:,1:M_N}} \in \mathbb{R}^{2(M-1)\times M_N}$; $\mathbf{T_{32}^*} = \mathbf{T^*_{3:,M_N+1:2(N-1)}} \in$ $\mathbb{R}^{2(M-1)\times(2(N-1)-M_N)}$ and λ, α, β and γ are parameters for LRP and proposed three variants of LRP, respectively; In addition, matrices D and U are defined in (9) of the main manuscript; matrices $\mathbf{T_1^*}$, $\mathbf{T_2^*}$ and $\mathbf{T_3^*}$ are defined in (12), (13) and (14) of the main manuscript, respectively.

The Jacobian matrix $\mathbf{J} = \frac{\partial \mathbf{q}}{\partial \mathbf{p}}$ can be calculated as follows:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f_A}}{\partial \dot{\delta}} & \frac{\partial \mathbf{f_A}}{\partial \dot{\eta}} & \cdots & \frac{\partial \mathbf{f_A}}{\partial \mathbf{w}} \\ \lambda\frac{\partial \mathbf{f_B}}{\partial \dot{\delta}} & \lambda\frac{\partial \mathbf{f_B}}{\partial \dot{\eta}} & \cdots & \lambda\frac{\partial \mathbf{f_B}}{\partial \mathbf{w}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta\frac{\partial \mathbf{f_E}}{\partial \dot{\delta}} & \beta\frac{\partial \mathbf{f_E}}{\partial \dot{\eta}} & \cdots & \beta\frac{\partial \mathbf{f_E}}{\partial \mathbf{w}} \end{bmatrix}. \tag{52}$$

Then the computation of block matrices in Eq. (52) can be expressed as follows:

$$\begin{cases} \frac{\partial \mathbf{f_A}}{\partial \dot{\delta}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \dot{\delta}_1}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \dot{\delta}_M}) \end{bmatrix} \\ \frac{\partial \mathbf{f_A}}{\partial \dot{\eta}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \dot{\eta}_2}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \dot{\eta}_N}) \end{bmatrix} \\ \frac{\partial \mathbf{f_A}}{\partial \mathbf{x}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \mathbf{X}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \mathbf{X}_{3,(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_A}}{\partial \mathbf{y}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \mathbf{Y}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \mathbf{Y}_{M_N,(2(N-1)-M_N)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_A}}{\partial \mathbf{z}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \mathbf{Z}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \mathbf{Z}_{(N-1+3),(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_A}}{\partial \mathbf{w}} = \begin{bmatrix} v(\frac{\partial \mathbf{U}}{\partial \mathbf{W}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{U}}{\partial \mathbf{W}_{(M-1+3),(M-1-3)}}) \end{bmatrix} \end{cases}, \tag{53}$$

where $\frac{\partial \mathbf{U}_{i-1,j-1}}{\partial \dot{\delta}_k} = \begin{cases} -2(\dot{\mathbf{t}}_{1,j} - \dot{\mathbf{t}}_{1,1}) + 2\dot{\eta}_j, & k = 1 \\ (2(\dot{\mathbf{t}}_{i,j} - \dot{\mathbf{t}}_{i,1}) - 2\dot{\eta}_j)\bullet \uparrow_{i,k}, & k = 2, \cdots, M \end{cases}$ and $\uparrow_{i,k}$ is represented by $\uparrow_{i,k} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$; $i = 2, \cdots, M$ and $j = 2, \cdots, N$, and $\frac{\partial \mathbf{U}_{i-1,j-1}}{\partial \dot{\eta}_k} = (-2(\dot{\mathbf{t}}_{i,j} - \dot{\mathbf{t}}_{1,j}) + 2(\dot{\delta}_1 - \dot{\delta}_i))\bullet \uparrow_{j,k}$ for $k = 2, \cdots, N$, and $\frac{\partial \mathbf{U}_{i-1,j-1}}{\partial \mathbf{X}_{k,l}} = 0$ for $k = 1, 2, 3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial U_{i-1,j-1}}{\partial Y_{k,l}} = 0$ for $k = 1, \cdots, M_N$ and $l = 1, \cdots, 2(N-1) - M_N$, and $\frac{\partial \mathbf{U}_{i-1,j-1}}{\partial \mathbf{Z}_{k,l}} = 0$ for $k = 1, \cdots, N-1+3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial \mathbf{U}_{i-1,j-1}}{\partial \mathbf{W}_{k,l}} = 0$ for $k = 1, \cdots, M-1+3$ and $l = 1, \cdots, M-1-3$.

Denote $\mathbf{V} = (\mathbf{A}+\mathbf{F})\mathbf{X} - \mathbf{B} - \mathbf{G}$, then

$$\begin{cases} \frac{\partial \mathbf{f_B}}{\partial \dot{\delta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \dot{\delta}_1}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \dot{\delta}_M}) \end{bmatrix} \\ \frac{\partial \mathbf{f_B}}{\partial \dot{\eta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \dot{\eta}_2}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \dot{\eta}_N}) \end{bmatrix} \\ \frac{\partial \mathbf{f_B}}{\partial \mathbf{x}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \mathbf{X}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \mathbf{X}_{3,(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_B}}{\partial \mathbf{y}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \mathbf{Y}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \mathbf{Y}_{M_N,2(N-1)-M_N}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_B}}{\partial \mathbf{z}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \mathbf{Z}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \mathbf{Z}_{N-1+3,N-1-3}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_B}}{\partial \mathbf{w}} = \begin{bmatrix} v(\frac{\partial \mathbf{V}}{\partial \mathbf{W}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V}}{\partial \mathbf{W}_{M-1+3,M-1-3}}) \end{bmatrix} \end{cases}, \tag{54}$$

where $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \dot{\delta}_1} = \sum_{u=1}^{3}(2(\dot{\eta}_{u+1} - \dot{\mathbf{t}}_{i,u+1} + \dot{\mathbf{t}}_{1,1})\mathbf{X}_{u,j-1}) - 2(\dot{\eta}_{j+3} - \dot{\mathbf{t}}_{i,j+3} + \dot{\mathbf{t}}_{1,1})$ for $i = 2, \cdots, M$ and $j = 2, \cdots, N-3$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \dot{\delta}_k} = \uparrow_{i,k} \bullet\{\sum_{u=1}^{3}(2(\dot{\mathbf{t}}_{i,u+1} - \dot{\mathbf{t}}_{i,1} - \dot{\eta}_{u+1})\mathbf{X}_{u,j-1}) - 2(\dot{\mathbf{t}}_{i,j+3} - \dot{\mathbf{t}}_{i,1} - \dot{\eta}_{j+3})\}$ for $k = 2, \cdots, M$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \dot{\eta}_k} = \begin{cases} (-2(\dot{\mathbf{t}}_{i,k} - \dot{\mathbf{t}}_{1,k}) + 2(\dot{\delta}_1 - \dot{\delta}_i))\mathbf{X}_{k-1,j-1} & k = 2, \cdots, 4 \\ (2(\dot{\mathbf{t}}_{i,j+3} - \dot{\mathbf{t}}_{1,j+3}) - 2(\dot{\delta}_1 - \dot{\delta}_i)) \bullet \uparrow_{k,j+3} & k = 5, \cdots, N \end{cases}$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \mathbf{X}_{k,l}} = \uparrow_{l,j-1} \bullet(\mathbf{A}+\mathbf{F})_{i-1,k}$ for $k = 1, 2, 3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \mathbf{Y}_{k,l}} = 0$ for $k = 1, \cdots, M_N$ and $l = 1, \cdots, 2(N-1) - M_N$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \mathbf{Z}_{k,l}} = 0$ for $k = 1, \cdots, N-1+3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial \mathbf{V}_{i-1,j-1}}{\partial \mathbf{W}_{k,l}} = 0$ for $k = 1, \cdots, M-1+3$ and $l = 1, \cdots, M-1-3$.

Denote $\mathbf{V1} = \mathbf{T}^*_{\mathbf{31}}\mathbf{Y} - \mathbf{T}^*_{\mathbf{32}}$, we have

$$\begin{cases} \frac{\partial \mathbf{f_C}}{\partial \dot{\delta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \dot{\delta}_1}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \dot{\delta}_M}) \end{bmatrix} \\ \frac{\partial \mathbf{f_C}}{\partial \dot{\eta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \dot{\eta}_2}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \dot{\eta}_N}) \end{bmatrix} \\ \frac{\partial \mathbf{f_C}}{\partial \mathbf{x}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \mathbf{X}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \mathbf{X}_{3,(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_C}}{\partial \mathbf{y}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \mathbf{Y}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \mathbf{Y}_{M_N,2(N-1)-M_N}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_C}}{\partial \mathbf{z}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \mathbf{Z}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \mathbf{Z}_{N-1+3,N-1-3}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_C}}{\partial \mathbf{w}} = \begin{bmatrix} v(\frac{\partial \mathbf{V1}}{\partial \mathbf{W}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V1}}{\partial \mathbf{W}_{M-1+3,M-1-3}}) \end{bmatrix} \end{cases}, \tag{55}$$

where $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \dot{\delta}_k} = \sum_{u=1}^{M_N}[\frac{\partial \mathbf{T}^*_{\mathbf{31}}}{\partial \dot{\delta}_k}]_{i-1,u}\mathbf{Y}_{u,j-1} - [\frac{\partial \mathbf{T}^*_{\mathbf{32}}}{\partial \dot{\delta}_k}]_{i-1,j-1}$ for $i = 2, \cdots, 2M-1$, $j = 2, \cdots, 2N-1-M_N$ and $k = 1, \cdots, M$, and $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \dot{\eta}_k} = (\sum_{u=1}^{M_N}[\frac{\partial \mathbf{T}^*_{\mathbf{31}}}{\partial \dot{\eta}_k}]_{i,u}\mathbf{Y}_{u,j} - [\frac{\partial \mathbf{T}^*_{\mathbf{32}}}{\partial \dot{\eta}_k}]_{i,j})$ for $k = 2, \cdots, N$, and $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \mathbf{X}_{k,l}} = \frac{\partial[\mathbf{T}^*_{\mathbf{31}}\mathbf{Y} - \mathbf{T}^*_{\mathbf{32}}]_{i-1,j-1}}{\partial \mathbf{X}_{k,l}} = 0$

for $k=1,\ 2,\ 3, l=1,\ \cdots,\ N-1-3$, and $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \mathbf{Y}_{k,l}} =\uparrow_{l,j-1} \bullet(\mathbf{T}^*_{\mathbf{31}})_{i-1,k}$, for $k=1,\ \cdots,\ M_N$ and $l=1,\ \cdots,\ 2(N-1)-M_N$, and $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \mathbf{Z}_{k,l}}=0$ for $k=1,\ \cdots,\ N-1+3$ and $l=1,\ \cdots,\ N-1-3$, and $\frac{\partial \mathbf{V1}_{i-1,j-1}}{\partial \mathbf{W}_{k,l}}=0$ for $k=1,\ \cdots,\ M-1+3$ and $l=1,\ \cdots,\ M-1-3$.

Denote $\mathbf{V2}=\mathbf{T}^*_{\mathbf{11}}\mathbf{Z}-\mathbf{T}^*_{\mathbf{12}}$, we have

$$\begin{cases} \frac{\partial \mathbf{f_D}}{\partial \dot{\delta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \dot{\delta}_1}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \dot{\delta}_M}) \end{bmatrix} \\ \frac{\partial \mathbf{f_D}}{\partial \dot{\eta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \dot{\eta}_2}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \dot{\eta}_N}) \end{bmatrix} \\ \frac{\partial \mathbf{f_D}}{\partial \mathbf{x}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \mathbf{X}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \mathbf{X}_{3,(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_D}}{\partial \mathbf{y}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \mathbf{Y}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \mathbf{Y}_{M_N,2(N-1)-M_N}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_D}}{\partial \mathbf{z}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \mathbf{Z}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \mathbf{Z}_{N-1+3,N-1-3}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_D}}{\partial \mathbf{w}} = \begin{bmatrix} v(\frac{\partial \mathbf{V2}}{\partial \mathbf{W}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V2}}{\partial \mathbf{W}_{M-1+3,M-1-3}}) \end{bmatrix} \end{cases}, \tag{56}$$

where $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \dot{\delta}_k}=\sum_{u=1}^{N-1+3}[\frac{\partial \mathbf{T}^*_{\mathbf{11}}}{\partial \dot{\delta}_k}]_{i-1,u}\mathbf{Z}_{u,j-1}-[\frac{\partial \mathbf{T}^*_{\mathbf{12}}}{\partial \dot{\delta}_k}]_{i-1,j-1}$ for $i=2,\ \cdots,\ M$, $j=2,\ \cdots,\ N-3$ and $k=1,\ \cdots,\ M$, and $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \dot{\eta}_k}=(\sum_{u=1}^{N-1+3}[\frac{\partial \mathbf{T}^*_{\mathbf{11}}}{\partial \dot{\eta}_k}]_{i-1,u} \mathbf{Z}_{u,j-1}-[\frac{\partial \mathbf{T}^*_{\mathbf{12}}}{\partial \dot{\eta}_k}]_{i-1,j-1})$ for $k=2,\ \cdots,\ N$, and $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \mathbf{X}_{k,l}}=0$ for $k=1,\ 2,\ 3$ and $l=1,\ \cdots,\ N-1-3$, and $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \mathbf{Y}_{k,l}}=0$ for $k=1,\ \cdots,\ M_N$ and $l=1,\ \cdots,\ 2(N-1)-M_N$, and $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \mathbf{Z}_{k,l}} =\uparrow_{l,j-1} \bullet(\mathbf{T}^*_{\mathbf{11}})_{i-1,k}$ for $k=1,\ \cdots,\ N-1+3$ and $l=1,\ \cdots,\ N-1-3$ and $\frac{\partial \mathbf{V2}_{i-1,j-1}}{\partial \mathbf{W}_{k,l}}=0$ for $k=1,\ \cdots,\ M-1+3$ and $l=1,\ \cdots,\ M-1-3$.

Denote $\mathbf{V3}=\mathbf{T}^*_{\mathbf{21}}\mathbf{W}-\mathbf{T}^*_{\mathbf{22}}$, we have

$$\begin{cases} \frac{\partial \mathbf{f_E}}{\partial \dot{\delta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \dot{\delta}_1}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \dot{\delta}_M}) \end{bmatrix} \\ \frac{\partial \mathbf{f_E}}{\partial \dot{\eta}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \dot{\eta}_2}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \dot{\eta}_N}) \end{bmatrix} \\ \frac{\partial \mathbf{f_E}}{\partial \mathbf{x}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \mathbf{X}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \mathbf{X}_{3,(N-1-3)}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_E}}{\partial \mathbf{y}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \mathbf{Y}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \mathbf{Y}_{M_N,2(N-1)-M_N}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_E}}{\partial \mathbf{z}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \mathbf{Z}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \mathbf{Z}_{N-1+3,N-1-3}}) \end{bmatrix} \\ \frac{\partial \mathbf{f_E}}{\partial \mathbf{w}} = \begin{bmatrix} v(\frac{\partial \mathbf{V3}}{\partial \mathbf{W}_{1,1}}) & \cdots & v(\frac{\partial \mathbf{V3}}{\partial \mathbf{W}_{M-1+3,M-1-3}}) \end{bmatrix} \end{cases}, \tag{57}$$

where $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \dot{\delta}_k}=\sum_{u=1}^{M-1+3}[\frac{\partial \mathbf{T}^*_{\mathbf{21}}}{\partial \dot{\delta}_k}]_{i-1,u}\mathbf{W}_{u,j-1}-[\frac{\partial \mathbf{T}^*_{\mathbf{22}}}{\partial \dot{\delta}_k}]_{i-1,j-1}$ for $i=2,\ \cdots,\ N$, $j=2,\ \cdots,\ M-3$ and $k=1,\ \cdots,\ M$, and $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \dot{\eta}_k}=(\sum_{u=1}^{M-1+3}[\frac{\partial \mathbf{T}^*_{\mathbf{21}}}{\partial \dot{\eta}_k}]_{i-1,u}$

$\mathbf{W}_{u,j-1} - [\frac{\partial \mathbf{T^*_{22}}}{\partial \dot{\eta}_k}]_{i-1,j-1})$ for $k = 2, \cdots, N$, and $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \mathbf{X}_{k,l}} = 0$ for $k = 1, 2, 3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \mathbf{Y}_{k,l}} = 0$ for $k = 1, \cdots, M_N$ and $l = 1, \cdots, 2(N-1) - M_N$, and $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \mathbf{Z}_{k,l}} = 0$ for $k = 1, \cdots, N-1+3$ and $l = 1, \cdots, N-1-3$, and $\frac{\partial \mathbf{V3}_{i-1,j-1}}{\partial \mathbf{W}_{k,l}} = \uparrow_{l,j-1} \bullet (\mathbf{T^*_{21}})_{i-1,k}$ for $k = 1, \cdots, M-1+3$ and $l = 1, \cdots, M-1-3$.

Table

Table 1: Summary of Notations

Symbol	**Description**	**Dimension**
M	Number of mics	Scalar
N	Number of acoustic sources	Scalar
$\mathbf{R}$	Matrix for mic position	$3 \times M$
$\mathbf{r}_i$	Vector for i^{th} mic position	3×1
$\mathbf{S}$	Matrix for sources position	$3 \times N$
$\mathbf{s}_j$	Vector for j^{th} sources position	3×1
$\mathbf{t}$	Asynchronous TOA matrix	$M \times N$
$\mathbf{t}_{i,j}$	Asynchronous TOA between i^{th} mic and j^{th} sources	Scalar
c	Speed of sound	Scalar
ζ	Asynchronous TDOA matrix	$(M-1) \times N$
$\zeta_{i,j}$	Asynchronous TDOA between i^{th} mic and j^{th} source	Scalar
δ	Vector for unknown start time of mic	$M \times 1$
δ_i	Unknown start time of i^{th} mic	Scalar
η	Vector for unknown emission time of source	$N \times 1$
η_j	Unknown emission time of j^{th} source	Scalar
δ'_i	Time offset between the i^{th} mic and 1^{st} mic	Scalar
$\hat{\eta}_j$	Negative value for synchronous TOA between 1^{st} mic and j^{th} source	Scalar

Continued on next page

Table 1: Summary of Notations (continued)

Symbol	Description	Dimension
$\hat{\delta}_i$	Negative value for time offset between the i^{th} mic and 1^{st} mic	Scalar
$\dot{\mathbf{t}}_{i,j}$	Transformation of asynchronous TOA between i^{th} mic and j^{th} source	Scalar
$\dot{\delta}_i$	Transformation of unknown start time for i^{th} mic	Scalar
$\dot{\eta}_j$	Transformation of unknown emission time for j^{th} source	Scalar
$\ddot{\zeta}_{i,j}$	Transformation of asynchronous TDOA between i^{th} mic and j^{th} source	Scalar
$\ddot{\delta}_i$	Transformation of unknown start time for i^{th} mic	Scalar
$\ddot{\eta}_j$	Transformation of unknown emission time for j^{th} source	Scalar
$\mathbf{D}$	Matrix for elements of asynchronous TOA	$(M-1)\times(N-1)$
$\mathbf{U}$	Matrix for element of asynchronous TOA and unknown time	$(M-1)\times(N-1)$
rank$(\cdot)$	Rank operator	Integer
$\mathbf{T_1}^*$	Elements for matrices $\mathbf{D}$ and $\mathbf{U}$	$(M-1)\times2(N-1)$
$\mathbf{T_2}^*$	Elements for matrices $\mathbf{D}$ and $\mathbf{U}$	$(N-1)\times2(M-1)$
$\mathbf{T_3}^*$	Elements for matrices $\mathbf{D}$ and $\mathbf{U}$	$2(M-1)\times2(N-1)$
min$\{\cdot\}$	Minimum operator	Scalar
$\mathbf{A}$	First three columns of matrix $\mathbf{D}$	$(M-1)\times3$
$\mathbf{B}$	Remaining columns of matrix $\mathbf{D}$	$(M-1)\times(N-1-3)$
$\mathbf{F}$	First three columns of matrix $\mathbf{U}$	$(M-1)\times3$

Continued on next page

Table 1: Summary of Notations (continued)

Symbol	Description	Dimension
$\mathbf{G}$	Remaining columns of matrix $\mathbf{U}$	$(M-1)\times(N-1-3)$
$\mathbf{X}$	Coefficient matrix for estimation	$3\times(N-1-3)$
$\mathbf{T_{11}}^*$	First $N-1+3$ columns of matrix $\mathbf{T_1}^*$	$(M-1)\times(N-1+3)$
$\mathbf{T_{12}}^*$	Remaining columns of matrix $\mathbf{T_1}^*$	$(M-1)\times(N-1-3)$
$\mathbf{Z}$	Coefficient matrix for estimation	$(N-1+3)\times(N-1-3)$
$\mathbf{T_{21}}^*$	First $M-1+3$ columns of matrix $\mathbf{T_2}^*$	$(N-1)\times(M-1+3)$
$\mathbf{T_{22}}^*$	Remaining columns of matrix $\mathbf{T_2}^*$	$(N-1)\times(M-1-3)$
$\mathbf{W}$	Coefficient matrix for estimation	$(M-1+3)\times(M-1-3)$
M_N	Defined as $M_N=\min\{N+2,\ M+2\}$	Scalar
$\mathbf{T_{31}}^*$	First M_N columns of matrix $\mathbf{T_3}^*$	$2(M-1)\times M_N$
$\mathbf{T_{32}}^*$	Remaining columns of matrix $\mathbf{T_3}^*$	$2(M-1)\times(2(N-1)-M_N)$
$\mathbf{Y}$	Coefficient matrix for estimation	$M_N\times(2(N-1)-M_N)$
f	Objective function	N/A
λ, α, β, γ	Penalty parameters	Scalars
C_1	Case label: $M-N>3$	N/A
C_2	Case label: $N-M>3$	N/A
C_3	Case label: $\lvert M-N\rvert\leq 3$	N/A
$v(\cdot)$	Column-wise matrix vectorization	N/A
$\mathbf{X}_{i,j}$	Element of i^{th} row and j^{th} column in matrix $\mathbf{X}$	Scalar

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Table 1: Summary of Notations (continued)

Symbol	Description	Dimension
Q	Dimension (length) of vectors $\mathbf{q}$	$(M-1)(8(N-1)-2M_N-6)-3(N-1)\times 1$
P	Dimension (length) of vector $\mathbf{p}$	$(M+N-1+3(N-1-3)+M_N(2(N-1)-M_N)+(N-1+3)(N-1-3)+(M-1+3)(M-1-3))\times 1$
$\mathbf{f_A}$	Column-wise matrix vectorization for matrix $\mathbf{U}$	$(M-1)(N-1)\times 1$
$\mathbf{f_B}$	Column-wise matrix vectorization for matrix $(\mathbf{A}+\mathbf{F})\mathbf{X}-\mathbf{B}-\mathbf{G}$	$(M-1)(N-1-3)\times 1$
$\mathbf{f_C}$	Column-wise matrix vectorization for matrix $\mathbf{T^*_{31}}\mathbf{Y}-\mathbf{T^*_{32}}$	$2(M-1)(2(N-1)-M_N)\times 1$
$\mathbf{f_D}$	Column-wise matrix vectorization for matrix $\mathbf{T^*_{11}}\mathbf{Z}-\mathbf{T^*_{12}}$	$(M-1)(N-1-3)\times 1$
$\mathbf{f_E}$	Column-wise matrix vectorization for matrix $\mathbf{T^*_{21}}\mathbf{W}-\mathbf{T^*_{22}}$	$(N-1)(M-1-3)\times 1$
$\mathbf{p}$	Vector for elements of $\dot{\delta}$, $\dot{\eta}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ and $\mathbf{W}$	$Q\times 1$
$\mathbf{q}$	Vector for elements of $\mathbf{f_A}$, $\mathbf{f_B}$, $\mathbf{f_C}$, $\mathbf{f_D}$ and $\mathbf{f_E}$	$P\times 1$
$\mathbf{J}$	Jacobian matrix	$Q\times P$
$Rr(M,N)$	Recovery rate	Scalar
$\sum$	Summation Operator	N/A
$Ne_i(M,N)$	Number of globally optimal solutions for i^{th} configuration	Integer
$I_n(M,N)$	Total number of configurations	Integer
$Nc(M,N)$	Total initializations for each configuration	Integer
er	Error metric for start time and emission time	Scalar

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Table 1: Summary of Notations (continued)

Symbol	**Description**	**Dimension**
$Cr(M,N)$	Convergence rate	Scalar
$Ce(M,N)$	Number of successful recoveries	Scalar
λ^*, α^*, β^*, γ^*	Penalty parameters	Scalars
$\mathcal{O}(\cdot)$	Computational Complexity	N/A
$\hat{J}_{1,r}$	Number of rows for J with LRP only	Integer
$\hat{J}_{1,c}$	Number of columns for J with LRP only	Integer
$\hat{J}_{1,4,r}$	Number of rows for J with LRP and LRPV3	Integer
$\hat{J}_{1,4,c}$	Number of columns for J with LRP and LRPV3	Integer
$\hat{J}_{1,2,4,r}$	Number of rows for J with LRP, LRPV1 and LRPV3	Integer
$\hat{J}_{1,2,4,c}$	Number of columns for J with LRP, LRPV1 and LRPV3	Integer
$\hat{J}_{1,2,3,r}$	Number of rows for J with LRP, LRPV2 and LRPV3	Integer
$\hat{J}_{1,2,3,c}$	Number of columns for J with LRP, LRPV2 and LRPV3	Integer
σ	Standard derivation	Scalar

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